Ex 1. Derive Bayes' rule:

$$p(H_i \mid X) = \frac{p(H_i, X)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X \mid H_i)\, p(H_i)}{p(X)} \quad \text{by P3}$$

$$= \frac{p(X \mid H_i)\, p(H_i)}{\sum_k p(X \mid H_k)\, p(H_k)} \quad \text{by rule of marginal prob.}$$

Ex 2: show $F \perp G \mid H \Rightarrow p(F \mid H, G) = p(F \mid H)$

(†) $p(F, G \mid H) = p(F \mid H)\, p(G \mid H)$ by definition

but also

(††) $p(F, G \mid H) = p(F \mid G, H)\, p(G \mid H)$ by P3

matching up (†) & (††):

$p(F \mid G, H)\, \cancel{p(G \mid H)} = p(F \mid H)\, \cancel{p(G \mid H)}$

<u>Support</u> set of values a r.v. can take.

$X \sim \text{binomial}(n, \theta)$

$X \in \{0, \ldots, n\}$

Exercise: identify the kernel

gamma kernel: $x^{\alpha-1} e^{-\beta x}$

Exercise. $\int_0^\infty x^{\alpha-1} e^{-\beta x} \, dx = ?$

$\frac{\Gamma(\alpha)}{\beta^\alpha}$

Law of total expectation

$$E \, \underbrace{E(X|\theta)}_{g(\theta)}$$
$$= \int \left[\int x \, p(x|\theta) \, dx \right] p(\theta) \, d\theta$$
$$= \int x \int \underbrace{p(x|\theta) \, p(\theta)} \, d\theta \, dx$$
$$= \int x \int p(x,\theta) \, d\theta \quad dx$$
$$= \int x \; p(x) \; dx \qquad \text{by rule of marginal prob.}$$
$$= EX \quad \square$$

Def'n <u>exchangeable</u> (subscripts don't matter)

Let $p(y_1, \ldots, y_n)$ be the joint density of $Y_1, \ldots, Y_n$. If $p(y_1, \ldots, y_n) = p(y_{\pi_1}, \ldots, y_{\pi_n})$ for all permutations π of $\{1, \ldots, n\}$ then $Y_1, \ldots, Y_n$ are <u>exchangeable</u>.

Ex 1: Urn with 2 red, 1 green

$$p(Y_1 = \text{red}, Y_2 = \text{green}) = p(Y_1 = \text{red}) \cdot p(Y_2 = \text{green} \mid Y_1 = \text{red})$$
$$= 2/3 \cdot 1/2$$
$$= 2/6 = 1/3$$

$$p(Y_1 = \text{green}, Y_2 = \text{red}) = p(Y_1 = \text{green}) \cdot p(Y_2 = \text{red} \mid Y_1 = \text{green})$$
$$= 1/3 \cdot 1$$
$$= 1/3$$

Y_1, Y_2 <u>are</u> exchangeable even though ~~not in~~dependent.

Ex 2.

coin 1 is a fair coin

coin 2 is double sided (heads only)

$Pr(Y_1 = H) = 0.5$

$Pr(Y_2 = H) = 1$

$p(0, 1) = 0.5$

$p(1, 0) = 0$

Y_1, Y_2 <u>are not</u> exchangeable.

Claim:

If $\theta \sim p(\theta)$ and $Y_1, \ldots, Y_n$ are conditionally iid given θ, then marginally (unconditional on θ) $Y_1, \ldots, Y_n$ are exchangeable.

Proof:

$$p(y_1, \ldots, y_n) = \int p(y_1, \ldots, y_n | \theta)\, p(\theta)\, d\theta$$ by rule of marginal prob.

$$= \int \left\{ \prod_{i=1}^{n} p(y_i | \theta) \right\} p(\theta)\, d\theta$$ by cond'l iid

$$= \int \left\{ \prod_{i=1}^{n} p(y_{\pi_i} | \theta) \right\} p(\theta)\, d\theta$$ products commute

$$= p(y_{\pi_1}, \ldots, y_{\pi_n})$$

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de Finetti's thm:

exchangeable $Y_1, \ldots, Y_n \quad \forall n$

$\Rightarrow \quad Y_1, \ldots, Y_n | \theta$ iid (for some parameter θ) and prior distribution $p(\theta)$.

- very cool because exchangeability is common!

$Y_1, \ldots, Y_n \rightarrow$ from repeatable experiment
- $\rightarrow$ sampled w/ replacement
- $\rightarrow \infty$ pop'n w/o replacement