$$X_i = \begin{cases} 1 & (H) \\ 0 & (T) \end{cases}$$

$$p(X_{100} = 1) \stackrel{?}{=} .5$$

$$P(X_{100} = 1 \mid X_1 = 0, X_2 = 0, \dots, X_{99} = 0) \begin{array}{l} \stackrel{?}{=} .5 \\ > .5 \\ < .5 \end{array}$$

$$X_i \cancel{\perp} X_j \quad i \neq j$$

But <u>exchangeable</u> seem plausible.

<u>de Finetti</u>

$\Rightarrow \quad X_i \perp X_j \mid \theta$ & identically distributed.

$$p(x_1, \dots, x_n \mid \theta) = p(x_1 \mid \theta) \cdot p(x_2 \mid \theta) \cdots p(x_n \mid \theta)$$
$$= \prod_{i=1}^{n} p(x_i \mid \theta)$$

What could θ be here?

θ : prob. coin lands heads. In which case:

$$p(x_i = 1 \mid \theta) = \theta$$
$$p(x_i = 0 \mid \theta) = (1 - \theta)$$

Together, $p(X_i = x_i \mid \theta) = \theta^{x_i} (1-\theta)^{1-x_i}$

<u>Ex</u>: write down the joint density:

$$\boxed{p(x_1, \dots, x_n \mid \theta)} = \prod_{i=1}^{n} p(x_i \mid \theta)$$
$$= \theta^{\Sigma x_i} (1 - \theta)^{n - \Sigma x_i}$$

$p(x_1, \dots, x_n \mid \theta)$ has 3 names:

(1) the joint density of the data

(2) "the data generative model"

(3) the likelihood function

Let $y = \Sigma x_i$ then

$$p(Y = y \mid \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

$$p(\theta) \longrightarrow p(\theta \mid y)$$

Bayes' thm. tells us how to update beliefs (about θ) w/ data.

"θ" is called a parameter of the data generative model.

likelihood

prior

$$\underbrace{p(\theta \mid y)}_{\boxed{\text{posterior}}} = \frac{\boxed{p(y \mid \theta)}\ \boxed{p(\theta)}}{\int_\theta p(y \mid \theta)\, p(\theta)\, d\theta}$$

normalizing const.

constant because

$$\int_\theta p(y, \theta)\, d\theta = p(y)$$

Ex: $\int_0^1 x^2\, dx = \frac{1}{3}$

What's a suitable prior $p(\theta)$?

What's the support of θ?

$$\theta \in [0,1]$$

beta & uniform(0,1) have correct support!

First, examining uniform(0,1)

prior $p(\theta) = \begin{cases} 1 & \text{if } \theta \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

likelihood: $p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$

"Bayesian Model"

$$p(\theta|y) = \frac{\binom{n}{y}\theta^y(1-\theta)^{n-y} \cdot \mathbb{1}\{\theta \in [0,1]\}}{c}$$

posterior

$$\propto \theta^{\overset{\alpha-1}{y}}(1-\theta)^{\overset{\beta-1}{n-y}} \cdot \mathbb{1}\{\theta \in [0,1]\}$$

This is the kernel of a beta(α, β)

where $\alpha = y+1$

$\beta = n-y+1$

show
$p(\theta|y)$ is a beta density.

$$\propto \underbrace{p(y|\theta)} \cdot \underbrace{p(\theta)}$$

$$\theta^{y}(1-\theta)^{n-y} \cdot \theta^{a-1}(1-\theta)^{b-1}$$

$$= \theta^{(y+a)-1}(1-\theta)^{(n+b-y)-1}$$

Let $\alpha = y+a$

$\beta = n+b-y$

then I have

$p(\theta|y) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ which is the kernel of a beta(α, β)

~~Ex~~

$$E\ \theta|y = \frac{a+y}{a+b+n}$$

$$\lim_{n\to\infty} \cancel{\frac{a}{a+b+n}} + \frac{y}{a+b+n}$$

0

Remember: $y = \Sigma X_i$

$= n\bar{x}$

where $\bar{x} = \frac{1}{n}\Sigma x_i$

$$= \lim_{n\to\infty} \frac{n\bar{x}}{a+b+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\bar{x}}{\cancel{\frac{a}{n}} + \cancel{\frac{b}{n}} + 1} = \bar{x}$$

$0 \quad 0$