"Completing the square"

Let $Y_i \mid \theta, \sigma^2 \overset{iid}{\sim} \text{normal}(\theta, \sigma^2)$ *note: $\vec{y} = \{y_1, \dots, y_n\}$

Let $\theta \mid \sigma^2 \sim \text{normal}(\mu_0, \tau_0^2)$

We want to prove that the full conditional posterior $\theta \mid \sigma^2, \vec{y}$ is normal with some posterior mean μ_n and posterior variance τ_n^2. i.e. we want to prove:

$$\boxed{\theta \mid \sigma^2, \vec{y} \sim \text{normal}(\mu_n, \tau_n^2)}$$

PROOF:

STEP 1: Bayes' thm:

$$p(\theta \mid \sigma^2, \vec{y}) \propto \underbrace{p(\vec{y} \mid \theta, \sigma^2)}_{\underbrace{\prod_{i=1}^{n} \text{dnorm}(y_i, \theta, \sigma)}_{\text{by iid}}} p(\theta \mid \sigma^2)$$

$$\propto \exp\left\{\frac{-1}{2\sigma^2} \Sigma (y_i - \theta)^2\right\} \cdot \exp\left\{\frac{-1}{2\tau_0^2}(\theta - \mu_0)^2\right\}$$

STEP 2: simplify by expanding quadratic terms and absorbing what we can into the normalizing constant. $\boxed{\text{*Note: } \Sigma y_i = n\bar{y}}$

$$\propto \exp\left\{\frac{-1}{2\sigma^2}(n\theta^2 - 2n\bar{y}\theta) - \frac{1}{2\tau_0^2}(\theta^2 - 2\mu_0\theta)\right\}$$

combine like terms:

$$\propto \exp\left\{-\frac{1}{2}\underbrace{\left(\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}\right)}_{\text{"a"}}\theta^2 + \underbrace{\left(\frac{n\bar{y}}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)}_{\text{"b"}}\theta\right\}$$

define "a" & "b":

$$\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta\right\}$$

(*) Notice: this is the kernel of a normal with $\boxed{\text{mean} = \frac{b}{a}}$ and $\boxed{\text{variance} = a^{-1}}$

therefore, $\theta \mid \sigma^2, \vec{y} \sim \text{normal}(\mu_n, \tau_n^2)$ where $\mu_n = \frac{b}{a}$ and $\tau_n^2 = a^{-1}$

and a, b are defined above. □

Proof of (*) on previous page: This answers the question:

How did I recognize $\exp\{-\frac{1}{2}a\theta^2 + b\theta\}$ as the kernel of a normal w/ mean $\frac{b}{a}$ and variance a^{-1}?

Assume $\theta \mid \vec{y}, \sigma^2 \sim \text{normal}\left(\frac{b}{a}, a^{-1}\right)$.

Then
$$p(\theta \mid \vec{y}, \sigma^2) \propto \exp\left\{-\frac{1}{2} a \left(\theta - \frac{b}{a}\right)^2\right\}$$
$$\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta \underbrace{-\frac{1}{2}\cancel{a}\frac{b^2}{a^{\cancel{2}}}}_{\text{constant in } \theta.}\right\}$$
$$\propto \exp\left\{-\frac{1}{2}a\theta^2 + b\theta\right\}$$