* Today's punchline:

Assume:

1. $Y_i \mid \theta, \sigma^2 \sim N(\theta, \sigma^2)$
2. $\theta \mid \sigma^2 \sim N(\mu_0, \sigma^2/\kappa_0)$
3. $1/\sigma^2 \sim \text{gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right)$

then the posterior

$$p(\theta, \sigma^2 \mid y_1, \ldots, y_n)$$

$$= \underbrace{p(\theta \mid \sigma^2, y_1, \ldots, y_n)}_{\text{dnorm}(\theta;\, \mu_n,\, \tau_n)} \; \underbrace{p(\sigma^2 \mid y_1, \ldots, y_n)}_{\text{dinvgamma}\left(\sigma^2;\, \frac{\nu_n}{2},\, \frac{\nu_n}{2}\sigma_n^2\right)}$$

"full cond'l posterior of θ"

Today's agenda:

(1) sketch proof for $p(\sigma^2 \mid y_1, \ldots, y_n)$

(2) sample from joint posterior

(3) sample from posterior predictive $p(\tilde{y} \mid y_1, \ldots, y_n)$

(time permitting)

Interpretation:

μ_0 : prior guess for θ

σ_0^2 : prior guess for σ^2

κ_0 : prior sample size for θ

ν_0 : prior sample size for σ^2

Assume

(1) $Y_i \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

(2) $\theta \mid \sigma^2 \sim N(\mu_0, \sigma^2 / \kappa_0)$

(3) $1/\sigma^2 \sim \text{gamma}\left(\nu_0/2, \frac{\nu_0 \sigma_0^2}{2}\right)$

then the joint posterior

$$p(\theta, \sigma^2 \mid \vec{y}) = p(\theta \mid \sigma^2, \vec{y})\, p(\sigma^2 \mid \vec{y})$$

$$= \text{dnorm}(\theta, \mu_n, \tau_n^2) \cdot \text{dinvgamma}\left(\sigma^2, \frac{\nu_n}{2}, \frac{\nu_n \sigma_n^2}{2}\right)$$

✓ shown previously

Goal: to prove $\sigma^2 \mid \vec{y} \sim$ invgamma

Sketch proof:

$$p(\sigma^2 \mid \vec{y}) \propto p(\vec{y} \mid \sigma^2)\, p(\sigma^2)$$

$$\propto p(\sigma^2) \int p(\vec{y}, \theta \mid \sigma^2)\, d\theta$$

$$\propto p(\sigma^2) \int p(\vec{y} \mid \theta, \sigma^2)\, p(\theta \mid \sigma^2)\, d\theta \qquad (\dagger)$$

Known by assumption: (3) (1) (2)

The integral $(\dagger)$ above reduces to integrating a normal density.

Explicit bookkeeping: attacking $\int p(\vec{y}\mid\theta,\sigma^2)\,p(\theta\mid\sigma^2)\,d\theta$

(1) collect (but do not absorb into normalizing const!) terms w/o θ outside integral:

$$\boxed{(2\pi\sigma^2)^{-n/2}\cdot\left(2\pi\sigma^2/K_0\right)^{-1/2}\cdot\exp\left\{\frac{-1}{2\sigma^2}\left(\Sigma y_i^2\right)-\frac{K_0}{2\sigma^2}(\mu_0)^2\right\}}\cdot \xrightarrow{} \text{call this "} C_1(\sigma^2) \text{"}$$

$$\cdot\int \exp\left\{\frac{-1}{2\sigma^2}\left(-2(\Sigma y_i)\theta+n\theta^2\right)-\frac{K_0}{2\sigma^2}\left(-2\mu_0\theta+\theta^2\right)\right\}d\theta$$

(2) Compute the integral above using the kernel trick:
hint: it's the kernel of a normal, you must complete the square.

$$\int e^{-\frac{1}{2}a\theta^2+b\theta}\,d\theta = \boxed{(2\pi/a)^{1/2}\exp\left\{\tfrac{1}{2}b^2/a\right\}} \xrightarrow{} \text{call this "} C_2(\sigma_2) \text{"}$$

where $a = \left(\frac{n}{\sigma^2}+\frac{K_0}{\sigma^2}\right)$

$b = \frac{(n\bar{y}+K_0\mu_0)}{\sigma^2}$

(3) Return to (†) and write down:

$$p(\sigma^2\mid\vec{y}) \propto p(\sigma^2)\cdot C_1(\sigma^2)\cdot C_2(\sigma^2)$$