What about nonconjugate priors?

$p(\theta \mid \vec{y}) \propto$ $\vec{y} = y_1, \ldots, y_n$

$$\underbrace{\prod_{i=1}^{n} \frac{\theta^{y_i} e^{-\theta}}{y_i!}}_{\substack{\text{likelihood} \\ p(y_1,\ldots,y_n \mid \theta)}} \cdot \underbrace{\frac{1}{\theta \sigma \sqrt{2\pi}} \exp\left\{ \frac{-(\ln\theta - \mu)^2}{2\sigma^2} \right\}}_{\substack{\text{Prior (log-normal)} \\ p(\theta)}}$$

$Y \mid \theta \sim \text{Poisson}$

$\theta > 0$

$$\mathbb{E}\, \theta \mid y_1, \ldots, y_n = \ ?$$

Exercise 1

$$p(\theta) = \frac{1}{2} \quad \boxed{0 < \theta < 2}$$

$$\phi = \log\theta \implies \theta = e^{\phi}$$

want $p(\phi)$

$$\rightarrow p(\phi)\, d\phi = p(\theta)\, d\theta$$

$$p(\phi) = \underbrace{p(\theta)}\underbrace{\left|\frac{d\theta}{d\phi}\right|}$$

$$p(\phi) = \frac{1}{2} e^{\phi} \qquad -\infty < \phi < \log 2$$

Exercise 2 ②

We want

$\vec{y}_2 = \{y_{21}, \ldots, y_{2n_2}\}$

$$p(\tilde{y}_1 < \tilde{y}_2 \mid \vec{y}_1, \vec{y}_2)$$

$\vec{y}_1 = \{y_{11}, \ldots, y_{1n_1}\}$

$\tilde{y}_1$: new obs of # children to a woman w/o bachelors

$\tilde{y}_2$: ← same but "w/ bachelors"

To compute w/ Monte Carlo sampling we need to be able to sample from

$$p(\tilde{y}_1 \mid \vec{y}_1, \vec{y}_2) \quad \text{and}$$

$$p(\tilde{y}_2 \mid \vec{y}_1, \vec{y}_2)$$

By independence assumed in our model:

$$p(\tilde{y}_1 \mid \vec{y}_1, \vec{y}_2) = p(\tilde{y}_1 \mid \vec{y}_1)$$

$$p(\tilde{y}_2 \mid \vec{y}_1, \vec{y}_2) = p(\tilde{y}_2 \mid \vec{y}_2)$$

In general, when $\tilde{y}$ is a new obs. from the same population, we call

$p(\tilde{y} \mid \vec{y})$ the "posterior predictive distribution"

$$\vec{Y}_1 \rightarrow \theta_1 \rightarrow \tilde{y}_1$$

$$\vec{Y}_2 \rightarrow \theta_2 \rightarrow \tilde{y}_2$$